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Uniform Bounds for Cardinal Hermite Spline Operators with Double Knots*

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A method is given for computing the uniform norm of the cardinal Hermite spline operator. This is the operator that takes two bounded biinfinite sequences of numbers into the unique bounded spline of degree $2k - 1 (k \geqslant 2)$ with knots of multiplicity two at the integers and that interpolates the two given sequences for both functional and first derivative values at the integers. The computational schema relies on knowledge of the Bernoulli splines, while the theoretical aspects make use of some properties of zeros of periodic splines.

I. INTRODUCTION

Given r bounded biinfinite sequences of real (or complex) numbers: $y^{(0)}, y^{(1)},..., y^{(r-1)}$, it is known [3] that there is a unique bounded spline function $S(x)$, with integer knots, of degree $2k - 1$ $(k \geqslant r)$ and continuity class C^{2k-1-r} that has the interpolating property

$$S^{(j)}(\nu) = y_\nu^{(j)}, \qquad \text{for} \quad j = 0, 1,..., r - 1, \quad \text{and all} \quad \nu \in Z.$$

We define L_{2k-1}^r as the operator that takes the vector $\bar{y} = (y^{(0)}, y^{(1)},..., y^{(r-1)})$ into the spline function $S(x)$. In [5], Richards gives a method for computing the norm of L_{2k-1}^1 when $\bar{y}$ and $S(x)$ each have the Chebyshev norm. We follow Richards' line of attack for L_{2k-1}^2. Thus, we are dealing with the cardinal Hermite interpolation problem. It is interesting to note that while Richards makes great use of the Euler splines, we are naturally forced to deal with the Bernoulli splines, Section III. In particular, a classical result concerning the continuity of the Bernoulli splines is needed.

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In addition, consideration must be given to periodic splines with double knots. We need to know about allowable zeros of such functions and this requires a slight extension (Section VI) of a theorem of Richards.

II. Definitions and Statement of the Problems

We let $\mathcal{S}_{m,r}$ denote the class of polynomial spline functions of degree m and continuity C^{m-r} at the integers. That is, $S \in \mathcal{S}_{m,r}$ if and only if

(i) S is a polynomial of degree at most m in every interval $(\nu, \nu + 1)$ and

(ii) $S \in C^{m-r}$ at the integers, that is, S and its first $m - r$ derivatives are continuous.

We consider $\bar{y} = (y^{(0)}, y^{(1)},\ldots, y^{(r-1)})$ to be a vector each of whose r components is a biinfinite sequence of real (or complex) numbers. We define the norm of $\bar{y}$ by

$$||| \bar{y} ||| = \max_{0 \leqslant j \leqslant r-1} \| y^{(j)} \|_\infty \tag{2.1}$$

where

$$\| y^{(j)} \|_\infty = \sup_\nu | y_\nu^{(j)} |. \tag{2.2}$$

If $||| \bar{y} ||| < \infty$, we know that for every $k \geqslant r$ there is a unique bounded $S(x) \in \mathcal{S}_{2k-1,r}$ that interpolates $\bar{y}$ in the sense that

$$S^{(j)}(\nu) = y_\nu^{(j)}, \qquad \text{for all} \quad \nu \in Z \quad \text{and} \quad 0 \leqslant j \leqslant r - 1. \tag{2.3}$$

This $S(x)$ may be written in the form (see [3]):

$$S(x) = \sum_\nu y_\nu^{(0)} L_0(x - \nu) + \sum_\nu y_\nu^{(1)} L_1(x - \nu) + \cdots + \sum_\nu y_\nu^{(r-1)} L_{r-1}(x - \nu), \tag{2.4}$$

where the $L_j(x) \in \mathcal{S}_{2k-1,r}$ are the fundamental interpolating functions having the property:

$$L_j^{(i)}(\nu) = \delta_{ij}\delta_{\nu 0}, \qquad \delta_{ij} \text{ is the Kronecker delta.} \tag{2.5}$$

We also know that the $L_j(x)$ decay exponentially as $| x | \to \infty$ and so the series in (2.4) converge locally uniformly.

We are interested in the operator L_{2k-1}^r defined on the space of bounded $\bar{y}$ as that operator that takes $\bar{y}$ into $S \in \mathcal{S}_{2k-1,r}$ defined by (2.4). Specifically, we are interested in the uniform norm of this operator:

$$\| L_{2k-1}^r \| = \sup_{|||\bar{y}||| \leqslant 1} \| L_{2k-1}^r \bar{y} \|_\infty .$$

If we define

$$\phi_j(x) = \sum_{\nu} | L_j(x - \nu)|, \tag{2.6}$$

then

$$\| L_{2k-1}^r \| = \sup_x \sum_{j=0}^{r-1} \phi_j(x). \tag{2.7}$$

We now point out that by definition (2.6), each $\phi_j(x)$ has period one. Hence, instead of searching for the right of (2.7), it is sufficient to look for

$$\max_{0 \leqslant x \leqslant 1} \sum_{j=0}^{r-1} \phi_j(x). \tag{2.8}$$

This is the problem we attack in the following sections for $r = 2$.

It will be important in these sections to know about $\mathscr{S}_{2k-1,2}^0$, the subspace of zero splines, and so we briefly review some material that first appeared in [3].

We say $S \in \mathscr{S}_{2k-1,2}^0$ if and only if $S \in \mathscr{S}_{2k-1,2}$ and $S^{(j)}(\nu) = 0$, for all $\nu \in Z$ and $j = 0, 1$. This space is easily seen to have dimension $2k - 4$. In [3], we showed there is a convenient basis for it.

We call $0 \neq S \in \mathscr{S}_{2k-1,2}^0$ an eigenspline with corresponding eigenvalue $\lambda \neq 0$ if

$$S(x + 1) = \lambda S(x), \qquad \text{for all } x. \tag{2.9}$$

In [3] we showed that there are $2k - 4$ independent eigensplines that form a basis for $\mathscr{S}_{2k-1,2}^0$. The corresponding eigenvalues are positive, simple, and appear in reciprocal pairs. That is, if λ is an eigenvalue, so is $1/\lambda$. In fact, the eigenvalues are the roots of the equation:

$$\begin{vmatrix} 1 & \binom{2}{1} & 1-\lambda & 0 & \cdots & & 0 \\ 1 & \binom{3}{1} & \binom{3}{2} & 1-\lambda & & 0 \quad \cdots & 0 \\ \vdots & & & & & & \vdots \\ 1 & \binom{n-2}{1} & \binom{n-2}{2} & & \cdots & \binom{n-2}{n-3} & 1-\lambda \\ 1 & \binom{n-1}{1} & \binom{n-1}{2} & & \cdots & & \binom{n-1}{n-2} \\ 1 & \binom{n}{1} & \binom{n}{2} & & \cdots & & \binom{n}{n-1} \end{vmatrix} = 0,$$

where, to simplify notation, we have written $n = 2k - 1$.

III. The Bernoulli Splines

In order to continue further, it is necessary to introduce the Bernoulli splines.

The nth Bernoulli polynomial, $B_n(x)$, is defined as the nth degree polynomial solution of the functional equation

$$f(x+1) - f(x) = nx^{n-1}, \qquad \text{for} \quad n = 1, 2, \ldots, \tag{3.1}$$

while $B_0(x) \equiv 1$. (Equivalently, $B_n(x)$ is defined as the coefficient of $u^n/n!$ in the identity $ue^{ux}/(e^u - 1) = \sum_{n=0}^{\infty} (u^n/n!)\, B_n(x)$. See [1].)

Using (3.1) it is easy to find

$$B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6}, \quad B_3(x) = x(x-1)(x-\tfrac{1}{2})$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \tfrac{1}{30}, \quad B_5(x) = x(x-1)(x-\tfrac{1}{2})(x^2 - x - \tfrac{1}{3}).$$

We define the normalized Bernoulli polynomial by

$$\beta_n(x) = B_n(x) - B_n(0) \tag{3.2}$$

and state without proof (see [1]) the following properties of $B_n(x)$ and $\beta_n(x)$.

$$B_n'(x) = nB_{n-1}(x), \qquad \text{for} \quad n \geqslant 1, \tag{3.3}$$

$$B_n(1-x) = (-1)^n B_n(x), \tag{3.4}$$

$$\beta_n(0) = \beta_n(1) = 0 = B_{2m-1}(0) = B_{2m-1}(1), \qquad \text{for} \quad n \geqslant 1, m \geqslant 2. \tag{3.5}$$

We now define the nth Bernoulli spline $\bar{\beta}_n(x)$, by

$$\begin{aligned} \bar{\beta}_n(x) &= \beta_n(x), && \text{if} \quad 0 \leqslant x \leqslant 1. \\ &= \beta_n(x - \nu), && \text{if} \quad \nu \leqslant x \leqslant \nu + 1. \end{aligned} \tag{3.6}$$

It is well known that $\bar{\beta}_n(x) \in \mathscr{S}_{n,2}$ for $n \geqslant 2$, since the periodic extension of $B_n(x)$ is the kernel for the Euler Maclaurin quadrature formula (cf. [3, Theorem 8, Corollary 2]).

IV. The Norm of L^2_{2k-1}

We return to the main discussion interrupted at the end of Section II. Throughout the sequel the degree remains $2k - 1$.

By specializing to $r = 2$ we gain information about the $\phi_j(x)$. In fact we have

THEOREM 1. *If $L_0(x)$ and $L_1(x)$ are the fundamental functions of interpolation in $\mathscr{S}_{2k-1,2}$ (see (2.5)), then*

$$| L_0(x)| = L_0(x), \qquad \textit{for all} \quad x \tag{4.1}$$

and

$$\begin{aligned} | L_1(x)| &= L_1(x), && \textit{if} \quad x \geqslant 0 \\ &= -L_1(x), && \textit{if} \quad x \leqslant 0. \end{aligned} \tag{4.2}$$

A proof of Theorem 1 appears in Section VI.

Thus, for $0 \leqslant x \leqslant 1$, we have

$$\phi_0(x) \equiv \sum_\nu | L_0(x - \nu)| = \sum_\nu L_0(x - \nu), \tag{4.3}$$

$$\phi_1(x) \equiv \sum_\nu | L_1(x - \nu)| = \sum_{\nu=-\infty}^{0} L_1(x - \nu) - \sum_{\nu=1}^{\infty} L_1(x - \nu). \tag{4.4}$$

We repeat that (4.3) and (4.4) are only valid for $0 \leqslant x \leqslant 1$. But the right side of (4.3), when x is allowed to vary over all of R, represents the unique bounded spline, $G(x) \in \mathscr{S}_{2k-1,2}$, which interpolates

$$\begin{aligned} \bar{y} &= y_\nu = 1 \\ &= y_\nu' = 0, \end{aligned}$$

for all ν. Hence,

$$G(x) \equiv 1. \tag{4.5}$$

Similarly the right side of (4.4) represents the unique bounded spline, $H(x) \in \mathscr{S}_{2k-1,2}$, which interpolates

$$\begin{aligned} \bar{y} = y_\nu &= 0, && \text{for all} \quad \nu \\ = y_\nu' &= -1, && \text{for} \quad \nu > 0 \\ &= 1, && \text{for} \quad \nu \leqslant 0. \end{aligned} \tag{4.6}$$

Then, in $0 \leqslant x \leqslant 1$,

$$\max_{0 \leqslant x \leqslant 1} \sum_{j=0}^{1} \phi_j(x) = \max_{0 \leqslant x \leqslant 1} [G(x) + H(x)] = 1 + \max_{0 \leqslant x \leqslant 1} H(x). \tag{4.7}$$

Thus, we are now seeking

$$\max_{0 \leqslant x \leqslant 1} H(x). \tag{4.8}$$

THEOREM 2. $\max_{0\leqslant x\leqslant 1} H(x) = H(\frac{1}{2})$.

The proof requires a discussion of periodic splines, and therefore, is postponed until Section VI.

COROLLARY. $\| L^2_{2k-1} \| = 1 + H(\frac{1}{2})$.

We now concern ourselves with constructing $H(x)$.

LEMMA. *$H(x)$ is symmetric about $x = \frac{1}{2}$.*

Proof. Consider $T(x) \in \mathscr{S}_{2k-1,2}$ defined by $T(x) = H(1 - x)$. Since $T(x)$ is bounded and interpolates (4.6), uniqueness of cardinal spline interpolation gives $T(x) = H(x)$. This proves the lemma.

We next consider the following normalized Bernoulli spline

$$\hat{\beta}(x) = \frac{\bar{\beta}_{2k-1}(x)}{(2k-1)\, B_{2k-2}}. \tag{4.9}$$

Then $\hat{\beta}(x) \in \mathscr{S}_{2k-1,2}$, and because of (3.3) and (3.5), $\hat{\beta}(x)$ interpolates

$$\begin{aligned} \bar{y} = y_\nu &= 0 \\ &= y_\nu' = 1, \end{aligned} \tag{4.10}$$

for all ν.

Notice that $H(x)$ interpolates (4.10) for $v \leqslant 0$. Let $\bar{H}(x)$ denote the unique extension (in $\mathscr{S}_{2k-1,2}$) of $H(x)$ (that is, of the restriction of $H(x)$ to $x \leqslant 0$), which interpolates (4.10) for all ν. Thus, $H(x) = \bar{H}(x)$ for $x \leqslant 0$.

Now

$$\bar{H}(x) - \hat{\beta}(x) \in \mathscr{S}^0_{2k-1,2},$$

and hence, there are unique $c_1, c_2, \ldots, c_{2k-4}$ such that

$$\bar{H}(x) = \hat{\beta}(x) + \sum_{i=1}^{2k-4} c_i S_i(x), \tag{4.11}$$

where the $S_i(x)$ are the eigensplines of $\mathscr{S}_{2k-1,2}$.

Since both $\bar{H}(x)$ and $\hat{\beta}(x)$ are bounded for $x \leqslant 0$, not all the eigensplines can appear in (4.11). Indeed, because of (2.9), only those whose corresponding eigenvalues are $\geqslant 1$ can appear. Since the eigenvalues appear in reciprocal pairs, exactly half appear in (4.11). Thus,

$$\bar{H}(x) = H(x) = \hat{\beta}(x) + \sum_{i=1}^{k-2} c_i S_i(x), \qquad \text{for} \quad x \leqslant 0. \tag{4.12}$$

To extend this representation of $H(x)$ to $x \leqslant 1$ we have only to add the proper truncated power functions:

$$H(x) = \hat{\beta}(x) + \sum_{i=1}^{k-2} c_i S_i(x) + \frac{a}{(2k-1)!} x_+^{2k-1} + \frac{b}{(2k-2)!} x_+^{2k-2}. \quad (4.13)$$

Since $H(x)$ is symmetric about $\frac{1}{2}$ and is just a polynomial in [0, 1], it must be that

$$H^{(j)}(\tfrac{1}{2}) = 0, \qquad \text{for} \quad j = 1, 3, 5, \ldots, 2k-1. \quad (4.14)$$

When (4.14) is applied to the representation (4.13), there results a system of k linear equations in the k unknowns: $\{c_i\}$, a, b. Since H is unique, this system has a unique solution, which, once found, is used to calculate $H(\frac{1}{2})$. Thus:

THEOREM 3.

$$H\left(\frac{1}{2}\right) =$$

$$\begin{vmatrix} \hat{\beta}\left(\frac{1}{2}\right) & S_1\left(\frac{1}{2}\right) & \cdots & S_{k-2}\left(\frac{1}{2}\right) & \frac{1}{(2k-1)!}\left(\frac{1}{2}\right)^{2k-1} & \frac{1}{(2k-2)!}\left(\frac{1}{2}\right)^{2k-2} \\ \hat{\beta}'\left(\frac{1}{2}\right) & S_1'\left(\frac{1}{2}\right) & \cdots & S_{k-2}'\left(\frac{1}{2}\right) & \frac{1}{(2k-2)!}\left(\frac{1}{2}\right)^{2k-2} & \frac{1}{(2k-3)!}\left(\frac{1}{2}\right)^{2k-3} \\ \hat{\beta}'''\left(\frac{1}{2}\right) & S'''\left(\frac{1}{2}\right) & \cdots & \vdots & \frac{1}{(2k-4)!}\left(\frac{1}{2}\right)^{2k-4} & \vdots \\ \vdots & \vdots & & \vdots & & \frac{1}{2} \\ \hat{\beta}^{(2k-1)}\left(\frac{1}{2}\right) & S_1^{(2k-1)}\left(\frac{1}{2}\right) & \cdots & S_{k-2}^{(2k-3)}\left(\frac{1}{2}\right) & 1 & 0 \end{vmatrix} \Delta_{11}^{-1}$$

where Δ_{11} is the minor of the leading element $\hat{\beta}(1/2)$.

V. NUMERICAL RESULTS FOR CUBIC AND QUINTIC INTERPOLATION

We now apply the results of the previous section to find $\| L_3{}^2 \|$ and $\| L_5{}^2 \|$.

(a) $k = 2$. In $\mathscr{S}^0_{3,2}$, there are no (nontrivial) elements, and hence, no eigensplines. The normalized Bernoulli spline (4.9) is

$$\hat{\beta}(x) = \beta_3(x)/3B_2 = 2x^3 - 3x^2 + x, \qquad \text{for} \quad 0 \leqslant x \leqslant 1.$$

(4.13) becomes

$$H(x) = 2x^3 - 3x^2 + x + (a/3!)\, x_+^3 + (b/2!)\, x_+^2, \qquad \text{for} \quad 0 \leqslant x \leqslant 1.$$

Applying (4.14) and solving gives

$$H(x) = -x^2 + x, \qquad \text{for} \quad 0 \leqslant x \leqslant 1,$$

and thus,

$$H(1/2) = 1/4.$$

By the corollary to Theorem 2,

$$\| L_3^2 \| = 5/4.$$

(b) $k = 3$. The eigenvalues for $\mathscr{S}_{5,2}^0$ are $\lambda = 3 \pm 8^{1/2}$. We use $\lambda_1 = 3 + 8^{1/2} > 1$. In $0 \leqslant x \leqslant 1$ its corresponding eigenspline is

$$S_1(x) = 2(1 + 2^{1/2})\, x^5 - (3 + 4 \cdot 2^{1/2})\, x^4 + 8^{1/2} x^3 + x^2.$$

The normalized Bernoulli spline (4.9) is

$$\hat{\beta}(x) = \beta_5(x)/5\beta_4 = -6x^5 + 15x^4 - 10x^3 + x, \qquad \text{for} \quad 0 \leqslant x \leqslant 1.$$

Applying (4.14) and solving the resulting system gives

$$\begin{aligned} a &= -(180/7)(8 + 9(2)^{1/2}), \\ b &= (6/7)(52 + 27(2)^{1/2}), \\ c_1 &= -(27/28)(2 - 3(2)^{1/2}), \end{aligned}$$

and therefore,

$$H(1/2) = (1/224)(41 + 54(2)^{1/2}) \approx 0.5239.$$

Thus, by the corollary to Theorem 2

$$\| L_5^2 \| \approx 1.5239.$$

VI. Periodic Splines and Proofs of Theorems 1 and 2

The proofs of Theorems 1 and 2 require some information about periodic splines with double knots. In particular, we are interested in the maximum number of zeros such functions can possess.

A function S is called n-periodic if $S(x + n) = S(x)$ for all x. We denote by $Z(S)_{[a,a+n]}$ the number of zeros of S on $[a, a + n]$ not counted according to multiplicities. A zero at the endpoints is counted just once, as is an interval with $S(x) \equiv 0$.

THEOREM 4 (cf. [5, Corollary 1 of Theorem 1]). *Let S be a 1-periodic spline with simple knots $0 = x_0 < x_1 < \cdots < x_n = 1$ on $[0, 1]$. Then $Z(S)_{[0,1]} \leqslant n$.*

For $0 < \epsilon < \frac{1}{2}$, we define the set A_ϵ by

$$A_\epsilon = \{i, i + \epsilon : i = 0, 1, \ldots, n - 1\} \cup \{n\}.$$

A slight generalization of the proof of Theorem 4 gives

COROLLARY 1. *Let S be an n-periodic spline with simple knots in A_ϵ. Then $Z(S)_{[0,n]} \leqslant 2n$.*

If we allow ϵ to approach 0 we have

COROLLARY 2. *Let S be an n-periodic spline in $S_{2k-1,2}$. Then $Z(S)_{[0,n]} \leqslant 2n$.*

We are now ready to prove Theorems 1 and 2.

Proof of Theorem 1. To prove (4.1), for $0 < \epsilon < \frac{1}{2}$ we consider the set

$$\begin{aligned} B_\epsilon &= \{0, 1, \ldots, n\} \cup \left\{1 + \epsilon, 2 + \epsilon, \ldots, \frac{n-1}{2} + \epsilon, \frac{n+1}{2} - \epsilon, \ldots, n - 1 - \epsilon\right\} \quad n \text{ odd} \\ &= \{0, 1, \ldots, n\} \cup \left\{1 + \epsilon, 2 + \epsilon, \ldots, \frac{n}{2} - 1 + \epsilon, \frac{n}{2} + 1 - \epsilon, \ldots, n - 1 - \epsilon\right\} \quad n \text{ even}, \end{aligned}$$

and define $L_{0,n}^\epsilon$ as the unique n-periodic spline of degree $2k - 1$ with simple knots in B_ϵ and satisfying $L_{0,n}^\epsilon(0) = L_{0,n}^\epsilon(n) = 1$, $L_{0,n}^\epsilon(t) = 0$ for all other $t \in B_\epsilon$. That $L_{0,n}^\epsilon$ exists and is unique follows from [4, Theorem 1.3]. By uniqueness it is clear that $L_{0,n}^\epsilon$ is symmetric about $x = 0$ and $x = n/2$. Finally, using the same proof as in [5, Lemma 1] we see that $L_{0,n}^\epsilon$ alternates in sign in each interval of $[0, n/2]$ bounded by two consecutive knots.

Since spline interpolation is a continuous function of the data, we have that $L_{0,n}^\epsilon$ converges to $L_{0,n} \in \mathscr{S}_{2k-1,2}$ as $\epsilon \to 0$, where $L_{0,n}$ is the n-periodic fundamental spline satisfying

$$L_{0,n}(\nu) = 1, \quad \text{for} \quad \nu \equiv 0 \pmod{n},$$

$$L_{0,n}(\nu) = 0, \quad \text{for} \quad \nu \not\equiv 0 \pmod{n},$$

and

$$L_{0,n}'(\nu) = 0, \quad \text{for all} \quad \nu \in Z.$$

We note that $L_{0,n}(x) \geqslant 0$ everywhere on $[0, n/2]$ since the intervals where $L^{\epsilon}_{0,n}$ was negative have contracted to points. Finally, since

$$L_{0,n}(x) = \sum_{i=-\infty}^{\infty} L_0(x - in),$$

we have

$$| L_0(x) - L_{0,n}(x)| \leqslant \sum_{i \neq 0} | L_0(x - in)|.$$

Since $L_0(x)$ decays exponentially, we get local uniform convergence of $L_{0,n}$ to L_0 as $n \to \infty$.

The proof of (4.2) is different. For even n, let $L_{1,n}(x)$ be the unique n-periodic element of $\mathscr{S}_{2k-1,2}$ satisfying

$$L_{1,n}(\nu) = 0, \qquad \text{for all} \quad \nu \in Z,$$

$$L'_{1,n}(\nu) = 1, \qquad \text{for} \quad \nu \equiv 0 \pmod{n},$$

and

$$L'_{1,n}(\nu) = 0, \qquad \text{for} \quad \nu \not\equiv 0 \pmod{n}.$$

Then $L_{1,n}$ is antisymmetric about $x = n/2$ by the uniqueness of spline interpolation. We claim

$$L_{1,n}(x) \geqslant 0, \qquad \text{for} \quad x \in [0, n/2]. \tag{6.7}$$

To show this we first assert that $L_{1,n}$ has zeros in $[0, n]$ only at $x = 0, 1,..., n$. Indeed $L'_{1,n}$ has a zero at $x = 1, 2,..., n - 1$ and by Rolle's theorem at least one zero in every $(\nu, \nu + 1)$ for $0 \leqslant \nu \leqslant n - 1$. If $L_{1,n}$ had an additional zero in $(r, r + 1)$ for $r < n/2$, this would contribute an extra zero of $L'_{1,n}$ in $(r, r + 1)$ and by antisymmetry an extra zero in $(n - r - 1, n - r)$ also. Thus, we would have at least $2n + 1$ zeros of $L'_{1,n}$ in $[0, n]$. This contradicts Corollary 2 of Theorem 4, and hence, $L_{1,n}$ has zeros only at the integers.

This established, the only way (6.7) can be violated is if there is an integer $\nu < (n/2) - 2$ such that $L_{1,n}$ changes sign at $\nu + 1$. A careful counting of the zeros of $L''_{1,n}$ in $[0, n]$ reveals, in this case, at least $2n + 2$ zeros. This again is a contradiction.

Finally, (4.2) follows by letting $n \to \infty$ and asserting that $L_{1,n}$ converges locally uniformly to L_1 as in the proof of (4.1) above.

Proof of Theorem 2. For any integer $N > 0$ consider

$$\begin{aligned} \bar{y}_N = y_\nu &= 0 \\ = y_\nu' &= -1, \qquad \text{if} \quad \nu = 1,..., N \\ &= 1, \qquad \text{if} \quad \nu = -N + 1,..., 0, \end{aligned}$$

and let $h_N(x)$ be the unique spline in $\mathscr{S}_{2k-1,2}$ defined on $[-N+1, N]$ that interpolates $\bar{y}_N$ and also satisfies the boundary conditions

$$h_N^{(j)}(-N+1) = -h_N^{(j)}(N), \qquad \text{for } j = 0, 1, \ldots, 2k-3.$$

Define

$$\begin{aligned} \tilde{h}_N(x) &= h_N(x), \qquad -N+1 \leqslant x \leqslant N \\ &= -h_N(x - 2N + 1), \qquad N \leqslant x \leqslant 3N - 1, \end{aligned} \tag{6.8}$$

and then extend $\tilde{h}_N(x)$ periodically on R (period $4N - 2$) and call the periodic extension $H_N(x)$. Clearly, $H_N(x) \in \mathscr{S}_{2k-1,2}$.

Then

$$| H(x) - H_N(x)| \leqslant 2 \sum_{|\nu| \geqslant N} | L_1(x - \nu)| \tag{6.9}$$

since $H - H_N \in \mathscr{S}_{2k-1,2}$ and interpolates

$$\begin{aligned} \bar{z} = z_\nu &= 0 \\ = z_\nu' &= 0, \qquad -N+1 \leqslant \nu \leqslant N \\ &= z_\nu', \qquad \text{otherwise}, \end{aligned}$$

where $| z_\nu' | \leqslant 2$.

Since $| L_1(x)| \to 0$ exponentially as $| x | \to \infty$, [3], we have from (6.9) that $H_N(x) \to H(x)$ uniformly on $[0, 1]$ as $N \to \infty$.

Hence, to prove the theorem it is sufficient to show

$$\max_{0 \leqslant x \leqslant 1} H_N(x) = H_N(\tfrac{1}{2}). \tag{6.10}$$

Indeed, in each of the intervals $[-N+1, 0]$, $[1, N]$, $[N, 2N-1]$, and $[2N, 3N-1]$ there are at least $2(N-1)+1$ distinct zeros of $H_N(x)$. Hence, by Rolle's theorem, there are at least $2(N-1)$ distinct zeros of $H_N'(x)$ in each of these intervals. Hence, a total of at least $8N - 8$ zeros in $[-N+1, 3N-1] - \{[0, 1] \cup [2N-1, 2N]\}$.

On the other hand, by Corollary 2 of Theorem 4, $H_N'(x)$ has at most $8N - 4$ zeros in $[-N+1, 3N-1]$. Hence, there are at most four zeros in $[0, 1]$ $[2N-1, 3N]$. By (6.8), each of these disjoint intervals has at most two zeros.

However, since $H_N(x)$ is symmetric about $x = \frac{1}{2}$ (same proof as Lemma 1), $H_N'(x)$ has an odd number of zeros in $[0, 1]$, one of which must be at $x = \frac{1}{2}$. Hence, $x = \frac{1}{2}$ is the only such zero. Finally, $H_N(\frac{1}{2})$ is a maximum since $H_N'(\frac{1}{2} - \epsilon) > 0$ and $H_N'(\frac{1}{2} + \epsilon) < 0$.

This proves (6.10) and hence, Theorem 2.

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